

UNCLASSIFIED

AD **405 054**

DEFENSE DOCUMENTATION CENTER

FOR

SCIENTIFIC AND TECHNICAL INFORMATION

CAMERON STATION, ALEXANDRIA, VIRGINIA



UNCLASSIFIED

Office of Naval Research

Contract Nonr-1866(32)

NR - 371 - 016

Technical Report

on

TABLES OF THE GENERALIZED STIRLING NUMBERS
OF THE FIRST KIND

by

William F. Pickard

March 1, 1963

The research reported in this document was supported by Grant 21869 of the National Science Foundation. Publication was made possible through support extended to Cruft Laboratory, Harvard University, by the Navy Department (Office of Naval Research), the Signal Corps of the U. S. Army, and the U. S. Air Force under ONR Contract Nonr-1866(32). Reproduction in whole or in part is permitted for any purpose of the United States Government.

Technical Report No. 404

Cruft Laboratory

Harvard University

Cambridge, Massachusetts

NOTICE: When government or other drawings, specifications or other data are used for any purpose other than in connection with a definitely related government procurement operation, the U. S. Government thereby incurs no responsibility, nor any obligation whatsoever; and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use or sell any patented invention that may in any way be related thereto.

TABLES OF THE GENERALIZED STIRLING NUMBERS
OF THE FIRST KIND

by

William F. Pickard*

Division of Engineering and Applied Physics
Gordon McKay Laboratory, Harvard University
Cambridge, Massachusetts

485054

 ABSTRACT

The generalized Stirling numbers of the first kind are defined, certain of their basic properties discussed, and tables given for the square grid $k = 0(1)10$ and $j = 0(1)10$ with $\ell = -10(1)10$. 

*The work reported in this paper was supported by a grant from the National Science Foundation.

The use of the losange diagram [1] to construct polynomial approximations, for interpolation, for the step by step integration of differential equations, for the derivation of formulae for numerical differentiation, and for other purposes, requires the evaluation of the factorial polynomials

$$(1) \quad (u - t)^{[k]} = (u - t)(u - t - 1) \cdots (u - t - k + 1)$$

where t and k are integers, k being non-negative and t unrestricted.

Since $(u - t)^{[k]}$ is a polynomial of degree k it can be represented as

$$(2) \quad \begin{aligned} (u - t)^{[k]} &= {}_t S_0^k u^k + {}_t S_1^k u^{k-1} + \cdots + {}_t S_k^k \\ &= \sum_{j=0}^k {}_t S_j^k u^{k-j} \end{aligned}$$

where the coefficients ${}_t S_j^k$ can be called the generalized Stirling numbers of the first kind in analogy with the terminology used for the numbers ${}_0 S_j^k$. The ${}_t S_j^k$ are conveniently determined using the recursive relationship

$$(3) \quad {}_t S_j^{k+1} = -(t+k) {}_t S_{j-1}^k + {}_t S_j^k \quad \begin{matrix} k = 0, 1, 2, \dots \\ j = 1, 2, 3, \dots \end{matrix}$$

and the obvious identities

$$(4a) \quad {}_t S_0^k = 1 \quad \begin{matrix} k = 0, 1, 2, \dots \end{matrix}$$

$$(4b) \quad {}_t S_j^0 = 0 \quad \begin{matrix} j = 1, 2, 3, \dots \end{matrix}$$

Equation (3) is readily derived by expanding

$$(5) \quad (u - l)^{[k+1]} = (u - l - k)(u - l)^{[k]}$$

and equating the coefficients of the several powers of u .

Only the ${}_l S_j^k$ for $l = 0$ have been extensively tabulated [2]; those for $l \neq 0$ appear to have been largely neglected. In order to facilitate the construction of formulas from the lozenge diagram, the IBM 7090 at the Harvard University Computing Center was utilized to calculate the ${}_l S_j^k$ over the square grid $k = 0(1)10$, $j = 0(1)10$ with $l = -10(1)10$. The results of these calculations are presented in Table I; a blank indicates that the number was zero and the word OVERFLOW that the absolute value of the number exceeded 34, 359, 738, 367.

REFERENCES

1. K. S. Kunz, "Numerical Analysis", McGraw-Hill Book Company, Inc., New York, 1957, chapter 4.
2. A. Fletcher, J. C. P. Miller, L. Rosenhead, and L. J. Comrie, "An Index of Mathematical Tables", Second Edition, Addison-Wesley Publishing Company, Inc., Reading, 1962, section 4.9231.

		$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$	$\{ = -10$
0	1	10	19	90	720	5040	30240	151200	604800	1814400	3628800
1	2	1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11	12	13
3	4	5	6	7	8	9	10	11	12	13	14
4	5	6	7	8	9	10	11	12	13	14	15
5	6	7	8	9	10	11	12	13	14	15	16
6	7	8	9	10	11	12	13	14	15	16	17
7	8	9	10	11	12	13	14	15	16	17	18
8	9	10	11	12	13	14	15	16	17	18	19
9	10	11	12	13	14	15	16	17	18	19	20
10	11	12	13	14	15	16	17	18	19	20	21

TABLE I GENERALIZED STIRLING NUMBERS OF THE FIRST KIND

		$\ell = -7$										
		$\ell = -6$										
		$\ell = -5$										
k	j	0	1	2	3	4	5	6	7	8	9	10
0	1	7	42	210	840	2754	5104	8028	5040	5040	5040	5040
-1	1	13	107	638	1175	245	1665	1960	6769	13132	13068	13068
2	1	18	22	179	245	2754	5104	13132	13068	13068	13068	13068
3	1	25	27	295	1665	1665	1665	1960	6769	13132	13068	13068
4	1	22	28	322	1960	1960	1960	1960	6769	13132	13068	13068
5	1	25	27	294	1638	1638	1638	1638	4809	6363	-64	-64
6	1	25	25	240	1533	1533	1533	1533	-3255	-12790	-7900	-7900
7	1											10000
8	1											11016
9	1											
10	1											
k	j	0	1	2	3	4	5	6	7	8	9	10
0	1	6	11	30	120	342	1064	360	720	720	720	720
-1	1	15	74	119	155	580	1624	1764	1764	1764	1764	1764
2	1	18	20	175	735	175	175	1624	1624	1624	1624	1624
3	1	21	21	175	175	175	175	1764	1764	1764	1764	1764
4	1	20	20	154	560	560	560	889	889	140	140	140
5	1	18	18	114	252	252	252	-231	-1638	-1324	-1324	-1324
6	1	15	15	60	-90	-90	-90	-987	-945	3590	3590	3590
7	1											-4320
8	1											-2664
9	1											
10	1											
k	j	0	1	2	3	4	5	6	7	8	9	10
0	1	5	9	20	47	60	154	120	120	120	120	120
-1	1	12	14	71	225	225	274	274	120	120	120	120
2	1	15	15	85	85	85	225	225	49	49	49	49
3	1	15	14	70	140	140	140	140	-154	-154	-154	-154
4	1	14	14	42	-231	-231	-231	-231	-252	-252	-252	-252
5	1	9	9	6	-126	-126	-126	-126	641	641	641	641
6	1	5	-30	-150	273	273	273	273	1365	1365	1365	1365
7	1								-820	-820	-820	-820
8	1								-4100	-4100	-4100	-4100
9	1								576	576	576	576
10	1											2680

TABLE I (CONTINUED)

TABLE I (CONTINUED)

		$\ell = -4$										
		0	1	2	3	4	5	6	7	8	9	10
k	j	0	1	4	12	26	50	24	-26	28	48	-144
		-2	1	9	26	50	24	-26	28	48	-144	-576
k	j	10	10	35	50	24	-26	28	48	-144	576	-2880
		10	10	25	15	-56	49	196	-36	48	-144	576
k	j	9	7	-35	-56	49	196	-36	48	-144	576	-2880
		4	-16	-56	273	-1365	-820	4100	4100	4100	4100	4100
k	j	-5	-30	150	273	-1365	-820	4100	4100	4100	4100	4100
		10	10	25	15	-56	49	196	-36	48	-144	576
		$\ell = -3$										
		0	1	2	3	4	5	6	7	8	9	10
k	j	0	1	3	5	6	6	-6	12	-36	144	-720
		-2	2	6	11	5	5	-5	49	-36	144	-720
k	j	3	6	11	5	5	5	-15	49	-36	144	-720
		3	-14	-14	56	49	49	-196	-36	144	-720	-2664
k	j	-4	-14	-14	126	-231	-441	944	324	144	-720	4320
		-9	6	6	126	-231	-441	945	3590	-5340	144	-720
k	j	-15	60	90	-967	945	3590	-5340	144	-720	4320	-2664
		10	10	25	15	-56	49	196	-36	48	-144	576
		$\ell = -2$										
		0	1	2	3	4	5	6	7	8	9	10
k	j	0	1	2	3	2	2	-2	4	-12	48	-240
		-2	2	3	3	2	2	-1	4	-12	48	-240
k	j	3	3	2	-5	15	4	-56	-28	188	-1368	1640
		3	-7	7	35	-231	252	-231	252	188	-1368	1640
k	j	-7	-12	42	-231	1638	-1324	1638	-1324	188	-1368	1640
		-12	114	-252	-231	1533	3255	-12790	3255	-12790	188	-1368
k	j	114	-252	-231	1533	3255	-12790	3255	-12790	188	-1368	1640
		240	-1050	-1050	1533	3255	-12790	3255	-12790	188	-1368	1640
k	j	10	10	25	15	-56	49	196	-36	48	-144	576

TABLE I (CONTINUED)

		$\ell = +5$										
k	j	0	1	2	3	4	5	6	7	8	9	10
0	1	-5	30	-210	1680	-11274	151200	151200	-1663200	19958400	-259459200	36322428500
1	2	-11	107	-1066	1125	-44524	-127860	-617624	8959148	-20355120	284574960	-4243508640
2	3	-18	251	-1066	1125	-13325	134449	-2231012	86837009	37972304	-136954044	2201931576
3	4	-26	485	-835	-8175	44524	-127860	-617624	8959148	-20355120	284574960	-4243508640
4	5	-35	1330	-11360	-33320	342769	-2231012	86837009	37972304	-136954044	2201931576	-4243508640
5	6	-45	2002	-2002	-59346	775929	-668566300	131590430	-668566300	2201931576	-4243508640	36322428500
6	7	-56	2886	-59346	1606773	-17550015	131590430	-668566300	2201931576	-4243508640	36322428500	
7	8	-68	4020	-99750								
8	9	-81										
9	10	-95										

		$\ell = +6$										
k	j	0	1	2	3	4	5	6	7	8	9	10
0	1	-6	42	-336	3024	-3024	-3024	332640	332640	-3991680	51891840	-726485760
1	2	-13	146	-1650	1924	-7524	-245004	3272688	-3991680	-46536624	51891840	-726485760
2	3	-21	325	-5000	1924	-7524	-245004	3272688	-3991680	-46536624	51891840	-726485760
3	4	-30	635	-2985	218344	-1139292	-3977164	18083484	-239705400	703604576	-726485760	
4	5	-40	1075	-1687	-29885	-41849	-197273	-11563650	73772180	-1406288100	5198985576	-1127754400
5	6	-51	2506	-46816	-81900	-197273	-29522145	297226930	-1406288100	5198985576	-1127754400	10897286400
6	7	-63	1687	-29885	-2506	-46816	-81900	-197273	-29522145	297226930	-1406288100	
7	8	-76	3570	-90	-135450	-25173	-25173	-25173	-25173	-25173	-25173	
8	9	-90	4920	-105								
9	10	-105										

		$\ell = +7$										
k	j	0	1	2	3	4	5	6	7	8	9	10
0	1	-7	56	-504	5040	-55440	-43468	665280	-8648640	-97053936	121080960	
1	2	-15	191	-2414	31594	-11754	-16815	-336049	-1961470	-6314664	-97053936	121080960
2	3	-24	431	-7155	31594	-11754	-16815	-336049	-1961470	-6314664	-97053936	121080960
3	4	-34	805	-805	-34300	81629	-6666516	33775244	-4721035	-436325840	-2743963940	11235811536
4	5	-45	1345	-1345	-34300	-63504	1768809	-18909891	133767584	-603682596	1576890000	-1816214400
5	6	-57	2086	-2086	-34300	-63504	1768809	-18909891	133767584	-603682596	1576890000	-1816214400
6	7	-70	3066	-3066	-108494	-108494	3520713	-4721035	-436325840	-2743963940	11235811536	-2704654400
7	8	-84	4326	-4326	-178710	-178710						
8	9	-99										
9	10	-115										

TABLE I (CONTINUED)

i	j	f_{i+j}
0	0	10
1	1	9
2	2	8
3	3	7
4	4	6
5	5	5
6	6	4
7	7	3
8	8	2
9	9	1
10	10	-8
11	11	-17
12	12	72
13	13	242
14	14	-720
15	15	539
16	16	-3382
17	17	920
18	18	-9850
19	19	4850
20	20	-95040
21	21	1235520
22	22	-72592
23	23	176554
24	24	-22785
25	25	1665
26	26	-63
27	27	2527
28	28	-45815
29	29	495544
30	30	-319348
31	31	11393808
32	32	-17297280
33	33	5935028
34	34	-18820400
35	35	259459200
36	36	-83720
37	37	1182769
38	38	-10630508
39	39	5935028
40	40	-18820400
41	41	259459200
42	42	-83720
43	43	1182769
44	44	-10630508
45	45	5935028
46	46	-18820400
47	47	259459200
48	48	-83720
49	49	1182769
50	50	-10630508
51	51	5935028
52	52	-18820400
53	53	259459200
54	54	-83720
55	55	1182769
56	56	-10630508
57	57	5935028
58	58	-18820400
59	59	259459200
60	60	-83720
61	61	1182769
62	62	-10630508
63	63	5935028
64	64	-18820400
65	65	259459200
66	66	-83720
67	67	1182769
68	68	-10630508
69	69	5935028
70	70	-18820400
71	71	259459200
72	72	-83720
73	73	1182769
74	74	-10630508
75	75	5935028
76	76	-18820400
77	77	259459200
78	78	-83720
79	79	1182769
80	80	-10630508
81	81	5935028
82	82	-18820400
83	83	259459200
84	84	-83720
85	85	1182769
86	86	-10630508
87	87	5935028
88	88	-18820400
89	89	259459200
90	90	-83720
91	91	1182769
92	92	-10630508
93	93	5935028
94	94	-18820400
95	95	259459200
96	96	-83720
97	97	1182769
98	98	-10630508
99	99	5935028
100	100	-18820400
101	101	259459200
102	102	-83720
103	103	1182769
104	104	-10630508
105	105	5935028
106	106	-18820400
107	107	259459200
108	108	-83720
109	109	1182769
110	110	-10630508
111	111	5935028
112	112	-18820400
113	113	259459200
114	114	-83720
115	115	1182769
116	116	-10630508
117	117	5935028
118	118	-18820400
119	119	259459200
120	120	-83720
121	121	1182769
122	122	-10630508
123	123	5935028
124	124	-18820400
125	125	259459200
126	126	-83720
127	127	1182769
128	128	-10630508
129	129	5935028
130	130	-18820400
131	131	259459200
132	132	-83720
133	133	1182769
134	134	-10630508
135	135	5935028
136	136	-18820400
137	137	259459200
138	138	-83720
139	139	1182769
140	140	-10630508
141	141	5935028
142	142	-18820400
143	143	259459200
144	144	-83720
145	145	1182769
146	146	-10630508
147	147	5935028
148	148	-18820400
149	149	259459200
150	150	-83720
151	151	1182769
152	152	-10630508
153	153	5935028
154	154	-18820400
155	155	259459200
156	156	-83720
157	157	1182769
158	158	-10630508
159	159	5935028
160	160	-18820400
161	161	259459200
162	162	-83720
163	163	1182769
164	164	-10630508
165	165	5935028
166	166	-18820400
167	167	259459200
168	168	-83720
169	169	1182769
170	170	-10630508
171	171	5935028
172	172	-18820400
173	173	259459200
174	174	-83720
175	175	1182769
176	176	-10630508
177	177	5935028
178	178	-18820400
179	179	259459200
180	180	-83720
181	181	1182769
182	182	-10630508
183	183	5935028
184	184	-18820400
185	185	259459200
186	186	-83720
187	187	1182769
188	188	-10630508
189	189	5935028
190	190	-18820400
191	191	259459200
192	192	-83720
193	193	1182769
194	194	-10630508
195	195	5935028
196	196	-18820400
197	197	259459200
198	198	-83720
199	199	1182769
200	200	-10630508
201	201	5935028
202	202	-18820400
203	203	259459200
204	204	-83720
205	205	1182769
206	206	-10630508
207	207	5935028
208	208	-18820400
209	209	259459200
210	210	-83720
211	211	1182769
212	212	-10630508
213	213	5935028
214	214	-18820400
215	215	259459200
216	216	-83720
217	217	1182769
218	218	-10630508
219	219	5935028
220	220	-18820400
221	221	259459200
222	222	-83720
223	223	1182769
224	224	-10630508
225	225	5935028
226	226	-18820400
227	227	259459200
228	228	-83720
229	229	1182769
230	230	-10630508
231	231	5935028
232	232	-18820400
233	233	259459200
234	234	-83720
235	235	1182769
236	236	-10630508
237	237	5935028
238	238	-18820400
239	239	259459200
240	240	-83720
241	241	1182769
242	242	-10630508
243	243	5935028
244	244	-18820400
245	245	259459200
246	246	-83720
247	247	1182769
248	248	-10630508
249	249	5935028
250	250	-18820400
251	251	259459200
252	252	-83720
253	253	1182769
254	254	-10630508
255	255	5935028
256	256	-18820400
257	257	259459200
258	258	-83720
259	259	1182769
260	260	-10630508
261	261	5935028
262	262	-18820400
263	263	259459200
264	264	-83720
265	265	1182769
266	266	-10630508
267	267	5935028
268	268	-18820400
269	269	259459200
270	270	-83720
271	271	1182769
272	272	-10630508
273	273	5935028
274	274	-18820400
275	275	259459200
276	276	-83720
277	277	1182769
278	278	-10630508
279	279	5935028
280	280	-18820400
281	281	259459200
282	282	-83720
283	283	1182769
284	284	-10630508
285	285	5935028
286	286	-18820400
287	287	259459200
288	288	-83720
289	289	1182769
290	290	-10630508
291	291	5935028
292	292	-18820400
293	293	259459200
294	294	-83720
295	295	1182769
296	296	-10630508
297	297	5935028
298	298	-18820400
299	299	259459200
300	300	-83720
301	301	1182769
302	302	-10630508
303	303	5935028
304	304	-18820400
305	305	259459200
306	306	-83720
307	307	1182769
308	308	-10630508
309	309	5935028
310	310	-18820400
311	311	259459200
312	312	-83720
313	313	1182769
314	314	-10630508
315	315	5935028
316	316	-18820400
317	317	259459200
318	318	-83720
319	319	1182769
320	320	-10630508
321	321	5935028
322	322	-18820400
323	323	259459200
324	324	-83720
325	325	1182769
326	326	-10630508
327	327	5935028
328	328	-18820400
329	329	259459200
330	330	-83720
331	331	1182769
332	332	-10630508
333	333	5935028
334	334	-18820400
335	335	259459200
336	336	-83720
337	337	1182769
338	338	-10630508
339	339	5935028
340	340	-18820400
341	341	259459200
342	342	-83720
343	343	1182769
344	344	-10630508
345	345	5935028
346	346	-18820400
347	347	259459200
348	348	-83720
349	349	1182769
350	350	-10630508
351	351	5935028
352	352	-18820400
353	353	259459200
354	354	-83720
355	355	1182769
356	356	-10630508
357	357	5935028
358	358	-18820400
359	359	259459200
360	360	-83720
361	361	1182769
362	362	-10630508
363	363	5935028
364	364	-18820400
365	365	259459200
366	366	-83720
367	367	1182769
368	368	-10630508
369	369	5935028
370	370	-18820400
371	371	259459200
372	372	-83720
373	373	1182769
374	374	-10630508
375	375	5935028
376	376	-18820400
377	377	259459200
378	378	-83720
379	379	1182769
380	380	-10630508
381	381	5935028
382	382	-18820400
383	383	259459200
384	384	-83720

i	j	$\ell = +9$	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell = 3$	$\ell = 4$	$\ell = 5$	$\ell = 6$	$\ell = 7$	$\ell = 8$	$\ell = 9$	$\ell = 10$	
1	1	-9	90	-990	11880	-4578	11880	-154440	71394	-115956	2162160	-32432400	OVERFLOW	
2	1	-19	299	-990	11880	-13145	71394	-115956	255424	-115956	19471500	-34397400	OVERFLOW	
3	1	-30	659	-4578	11880	-1205	71394	-115956	3010	-983316	16223700	-230997852	OVERFLOW	
4	1	-42	659	-4578	11880	-13015	71394	-115956	-59640	705649	-16223700	-636651200	OVERFLOW	
5	1	-55	1205	-13145	71394	-1975	255424	-115956	-107800	165988	-9923556	-8821612800	OVERFLOW	
6	1	-69	1975	-30015	255424	-115956	-115956	-181818	3492489	-44493813	375923456	-8797620060	OVERFLOW	
7	1	-84	3010	-983316	16223700	-100	3354	-16223700	-290790	6765213	-107358615	1176812090	-8797620060	OVERFLOW
8	1	-100	4354	-16223700	16223700	-117	6054	-16223700	-290790	6765213	-107358615	1176812090	-8797620060	OVERFLOW
9	1	-117	6054	-181818	3492489	-135	8160	-44493813	-290790	6765213	-107358615	1176812090	-8797620060	OVERFLOW
10	1	-135	8160	-290790	6765213	-107358615	1176812090	-8797620060					JVERFL 34	

TABLE I (CONTINUED)

